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MINIMAL INVARIANT FUNCTIONS OF THE SPACE-TIME WIENER PROCESS

BY

KAI YUEN WOO

ABSTRACT. Minimal invariant functions of the space-time Wiener process are obtained.

1. **Introduction.** Robbins and Siegmund [9] obtained an integral representation of any nonnegative invariant function of the space-time Wiener process as

$$h(x,t) = \int_{-\infty}^{\infty} \exp\left(\lambda x - \frac{\lambda^2 t}{2}\right)\mu(d\lambda)$$

for some measure μ on $(-\infty,\infty)$. These functions are also the nonnegative c^∞-solutions of $\partial h/\partial t + \frac{1}{2}(\partial^2 h/\partial x^2) = 0$, $(x,t) \in (-\infty,\infty)\times(0,\infty)$ such that $\lim_{t\to 0} h(x,t) = h(x,0)$ exists as a function, cf. Lai [6], McKean [7]. However, such a representation also falls into general considerations in Martin boundary theory, cf. Meyer [8]. It is the purpose of this paper to establish by elementary methods that the functions $\exp(\lambda x - \lambda^2 t/2)$ are minimal invariant functions of the space-time Wiener process and later on to establish the above representation by Martin boundary theory. On the other hand, Doob, Snell and Williamson [4] constructed minimal invariant functions of random walks on the N-dimensional lattice. Our result will offer an example in the continuous time case.

2. **Definitions and main results.** We shall follow Dynkin [5] in the definition and notation of a Markov process. Let $X' = (x_t^1, \zeta^1, \mathfrak{M}_t^1, P_{x^1}^1)$ be a Wiener process on the real line R with almost all sample functions continuous. Let $X^2 = (x_t^2, \zeta^2, \mathfrak{M}_t^2, P_{x^2}^2)$ be the process of uniform motion to the right on $[0,\infty)$ with transition function $P(t,x^2,T) = \chi_T(x^2+t)$ $(x^2 \in [0,\infty))$. X^1 and X^2 are Markov processes and their joint process $X = (X^1, X^2) = (x_t, \zeta, \mathfrak{M}_t, P_x)$ is a Markov process on $R \times [0,\infty)$ with transition function $P(t,x,A)$ generated

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by assignments on measurable rectangles of $\mathfrak{B}(\mathfrak{R}) \times \mathfrak{B}([0,\infty))$ given by

$$P(t,(x^1,x^2),\Gamma \times T) = \chi_T(x^2+t)\int_\Gamma \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x^1)^2}{2t}\right)dy$$

where we write (x^1,x^2) for x. This joint process X we shall call the space-time Wiener process. It is a Markov process with almost all sample functions continuous. For convenience we denote the state space $(R\times[0,\infty), \mathfrak{B}(\mathfrak{R}) \times \mathfrak{B}([0,\infty)))$ of X by $(E,\mathfrak{B})$ and any point in E by (x, u). A nonnegative measurable function h on $(E,\mathfrak{B})$ is said to be *invariant* (for X) if

$$P_t h = h \quad \text{for all } t \geqslant 0$$

where

$$P_t h(x,u) = \int_E P(t,(x,u),d(y,v))h(y,v) \qquad ((x,u)\in E).$$

An invariant function h is said to be *minimal* if for any other invariant function g such that $0 \leqslant g \leqslant h$ we have $g = ch$ for some constant c. The main theorems obtained in this paper are the following:

THEOREM 1. *Let h be a nonnegative measurable function on $(E,\mathfrak{B})$ which satisfies $h(0,0) = 1$. Then h is a minimal invariant function if and only if the following two conditions are satisfied*:

(a) $P_t h(0,0) = 1$ *for all* $t \geqslant 0$,

(b) $h[(x,u)+(y,v)] = h(x,u)h(y,v)$ *for all* $(x,u), (y,v) \in E$.

THEOREM 2. *The minimal invariant functions h for the space-time Wiener process which satisfy $h(0,0) = 1$ are* $\exp(\alpha x - \alpha^2 u/2)$ $((x,u)\in E)$ *where α runs through* $(-\infty,\infty)$.

The rest of the paper will be devoted to proofs of the above theorems.

3. **Proof of Theorem 1.** We first prove the necessity part of Theorem 1. We shall denote by $p(t, x, \Gamma)$ the transition function of the Wiener process on the real line R.

LEMMA 1. *If h is a nonnegative measurable function on $(E,\mathfrak{B})$, then for any $t \geqslant 0$,*

(a) $$P_t h(x,u) = \int_R p(t,x,dy)h(y,u+t) \qquad ((x,u)\in E),$$

and

(b) $$P_t h(x+y,u+v) = \int_E P(t,(x,u),d(z,w))h[(z,w)+(y,v)] \qquad ((x,u),(y,v)\in E).$$

PROOF.

(a)
$$\begin{aligned} P_t h(x,u) &= \int_E P(t,(x,u),d(y,v))h(y,v) \\ &= \int_{R\times\{u+t\}} P(t,(x,u),d(y,v))h(y,v) \\ &= \int_R p(t,x,dy)h(y,u+t). \end{aligned}$$

(b)
$$\begin{aligned} P_t h(x+y,u+v) &= \int_E P(t,(x+y,u+v),d(z,w))h(z,w) \\ &= \int_R p(t,x+y,dz)h(z,u+v+t) \\ &= \int_R p(t,x,dz)h(y+z,u+v+t) \\ &= \int_E P(t,(x,u),d(z,w))h(y+z,v+w) \\ &= \int_E P(t,(x,u),d(z,w))h[(z,w)+(y,v)]. \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 2. *If h is an invariant function, $h(0,0)=1$, then h is positive everywhere and for each $u>0$, $h(x,u)$ is finite and is a continuous function in x.*

PROOF. (a) Suppose $h(x_0,u_0)=0$ for some $(x_0,u_0)\in E$. By the invariant property we have for any $t>0$, $h(y,u_0+t)=0$ for almost every y on R, which by the invariant property again implies $h(0,0)=0$, a contradiction.

(b) Let $u>0$ be arbitrarily fixed. For any $t>0$,

$$\begin{aligned} h(x,u) &= \int_R p(t,x,dy)h(y,u+t) \\ &= \int_R \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x)^2}{2t}\right)h(y,u+t)\,dy \\ &= \int_R \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x)^2}{2t}\right)\sqrt{2\pi(u+t)}\exp\left(\frac{y^2}{2(u+t)}\right) \\ &\qquad \times \frac{1}{\sqrt{2\pi(u+t)}}\exp\left(-\frac{y^2}{2(u+t)}\right)h(y,u+t)\,dy \\ &= \int_R \sqrt{\frac{u+t}{t}}\exp\left(-\frac{uy^2-2(u+t)xy+(u+t)x^2}{2t(u+t)}\right) \\ &\qquad \times \frac{1}{\sqrt{2\pi(u+t)}}\exp\left(-\frac{y^2}{2(u+t)}\right)h(y,u+t)\,dy. \end{aligned}$$

Since

$$\int_R \frac{1}{\sqrt{2\pi(u+t)}} \exp\left(-\frac{y^2}{2(u+t)}\right) h(y, u+t)\, dy = h(0,0) = 1$$

and $u > 0$, $h(x, u)$ is finite for all $x \in R$, and by the dominated convergence theorem we have $\lim_{x \to x_0} h(x, u) = h(x_0, u)$. Q.E.D.

LEMMA 3. *Let h be an invariant function, $h(0,0) = 1$. Let $(y_0, v_0) \in E$, $v_0 > 0$, be arbitrarily fixed. Let $U_n = \{(y, v) \in E : |y - y_0| < 1/n\}$ and μ_{v_0} be the measure on $(E, \mathfrak{B})$ such that*

$$\mu_{v_0}(A) = \int_A P(v_0, (0,0), d(y,v)) h(y,v) \qquad (A \in \mathfrak{B}).$$

Then

(a) $0 < \mu_{v_0}(U_n) < \infty$ $(n = 1, 2, \ldots)$,

(b) *the functions*

$$h_n(x,u) = \int_{U_n} \frac{h[(x,u) + (y,v)]}{h(y,v)} \mu_{v_0}(d(y,v))$$

are invariant, and

(c) $0 \leqslant h_n \leqslant h$.

PROOF. (a)

$$\begin{aligned}\mu_{v_0}(U_n) &= \int_{U_n} P(v_0, (0,0), d(y,v)) h(y,v) \\ &= \int_{(y_0 - 1/n, y_0 + 1/n)} p(v_0, 0, dy) h(y, v_0) \\ &= \int_{(y_0 - 1/n, y_0 + 1/n)} \frac{1}{\sqrt{2\pi v_0}} \exp\left(-\frac{y^2}{2v_0}\right) h(y, v_0)\, dy.\end{aligned}$$

Since by Lemma 2, $h(y, v_0)$ is positive and continuous in y, the result follows.

(b) and (c) follow from Lemma 1(b). Q.E.D.

To prove the necessity part of Theorem 1 we assume that h is a minimal invariant function on $(E, \mathfrak{B})$ with $h(0,0) = 1$. We shall show that (a) and (b) of Theorem 1 are satisfied.

For (a), $P_t h(0,0) = h(0,0) = 1$ for all $t \geqslant 0$.

For (b), let $(y_0, v_0) \in E$, $v_0 > 0$, be arbitrarily fixed and let U_n, μ_{v_0} and $h_n(x, u)$ be as defined in Lemma 3. By the same lemma the functions h_n are invariant and satisfy $0 \leqslant h_n \leqslant h$. By the minimality of h we have $h_n = c_n h$ for some constants c_n. $c_n = h_n(0,0) = \mu_{v_0}(U_n)$ because $h(0,0) = 1$. Thus

$$h(x,u) = \frac{1}{\mu_{v_0}(U_n)} \int_{U_n} \frac{h[(x,u)+(y,v)]}{h(y,v)} \mu_{v_0}(d(y,v))$$

$$= \frac{1}{\mu_{v_0}(U_n)} \int_{U_n} \frac{h[(x,u)+(y,v_0)]}{h(y,v_0)} \mu_{v_0}(d(y,v)).$$

The continuity of $h(y,v_0)$ in y implies that as $n \to \infty$, the right-hand side of the above equation tends to $h[(x,u)+(y_0,v_0)]/h(y_0,v_0)$. The left-hand side being independent of n, we have

$$h(x,u) = h[(x,u)+(y_0,v_0)]/h(y_0,v_0).$$

Since $(y_0,v_0) \in E$ is arbitrary, subject only to the condition $v_0 > 0$,

$$h[(x,u)+(y,v)] = h(x,u)h(y,v)$$

holds for all (x,u), $(y,v) \in E$ such that not both u and v are zero. If $u = v = 0$, then for any $t > 0$,

$$\begin{aligned} h((x,0)+(y,0)) &= h(x+y,0) = P_t h(x+y,0) \\ &= \int_E P(t,(x,0),d(z,w))h[(z,w)+(y,0)] \\ &= \int_R p(t,x,dz)h[(z,t)+(y,0)] = \int_R p(t,x,dz)h(z,t)h(y,0) \\ &= \left[\int_E P(t,(x,0),d(z,w))h(z,w)\right]h(y,0) \\ &= h(x,0)h(y,0). \end{aligned}$$

Thus the necessity part of Theorem 1 is proved.

Next we prove the sufficiency part of Theorem 1. Throughout the rest of this section we shall assume that h is a nonnegative measurable function on $(E,\mathfrak{B})$ such that $h(0,0) = 1$ and conditions (a) and (b) of Theorem 1 are satisfied. We shall show that h is a minimal invariant function.

LEMMA 4. *h is an invariant function, finite and positive everywhere on E.*

PROOF. (a) For any $t \geqslant 0$, by Lemma 1,

$$\begin{aligned} P_t h(x,u) &= P_t h(0+x,0+u) \\ &= \int_E P(t,(0,0),d(z,w))h[(z,w)+(x,u)] \\ &= \int_E P(t,(0,0),d(z,w))h(z,w)h(x,u) \\ &= [P_t h(0,0)]h(x,u) = h(x,u). \end{aligned}$$

(b) From Lemma 2 and above we see that $h(x,u)$ is finite and positive for all $(x,u) \in E$ such that $u > 0$. If $u > 0$, then for any $v > 0$,

$$h(x,0)h(0,v) = h(x+0, 0+v) = h(x,v)$$

for any x. Since $h(x,v)$ and $h(0,v)$ are finite and positive, $h(x,0)$ is finite and positive also. Q.E.D.

LEMMA 5. *The function* $q(t,x,\Gamma)$ $(t \geqslant 0, x \in R, \Gamma \in \mathfrak{B}(R))$, *defined as*

$$q(t,x,\Gamma) = \frac{1}{h(x,0)} \int_\Gamma p(t,x,dy)h(y,t),$$

is a transition function on $(R, \mathfrak{B}(R))$ *with* $q(t,x,R) = 1$.

PROOF. $q(t,x,\Gamma)$ is well defined because of Lemma 4. For each $t \geqslant 0$, it is easy to see that $q(t,x,\cdot)$ is a measure on $\mathfrak{B}(R)$ for each $x \in R$ and $q(t,\cdot,\Gamma)$ is a $\mathfrak{B}(R)$-measurable function for each $\Gamma \in \mathfrak{B}(R)$. $q(t,x,R) = 1$ because

$$\begin{aligned} q(t,x,R) &= \frac{1}{h(x,0)} \int_R p(t,x,dy)h(y,t) \\ &= [1/h(x,0)][P_t h(x,0)] = h(x,0)/h(x,0) = 1. \end{aligned}$$

Finally,

$$\begin{aligned} &\int_R q(s,x,dy)q(t,y,\Gamma) \\ &\quad = \int_R \frac{1}{h(x,0)} p(s,x,dy)h(y,s) \frac{1}{h(y,0)} \int_\Gamma p(t,y,dz)h(z,t) \\ &\quad = \frac{h(0,s)}{h(x,0)} \int_R p(s,x,dy) \int_R p(t,y,dz)\chi_\Gamma(z)h(z,t) \\ &\quad = \frac{h(0,s)}{h(x,0)} \int_R p(s+t,x,dz)\chi_\Gamma(z)h(z,t) \\ &\quad = \frac{1}{h(x,0)} \int_\Gamma p(s+t,x,dz)h(z,s+t) \\ &\quad = q(s+t,x,\Gamma). \quad \text{Q.E.D.} \end{aligned}$$

By a general theorem (see [5]) there exists a Markov process $Y^1 = (y_t^1, \zeta^1, \mathfrak{M}_t^1, P_y^1)$ on $(R, \mathfrak{B}(R))$ which has $q(t,x,\Gamma)$ as its transition function. Moreover we can regard $\zeta^1 \equiv \infty$ because $q(t,x,R) = 1$. If we consider the joint process of Y^1 with the process of uniform motion to the right we obtain a Markov process $Y = (y_t, \zeta, \mathfrak{M}_t, P_y)$ on $E = R \times [0,\infty)$ with $\zeta \equiv \infty$. Its transition function $Q(t,(x,u),A)$ is given by

$$Q(t,(x,u),A) = \frac{1}{h(x,u)} \int_A P(t,(x,u),d(y,v))h(y,v)$$

for every $A \in \mathcal{B}$.

LEMMA 6. *If r is any nonnegative measurable function on $(E,\mathcal{B})$, then for every $t \geqslant 0$, $(x,u) \in E$,*

$$\int_E Q(t,(x,u),d(y,v))r(y,v) = \int_E Q(t,(0,0),d(y,v))r[(x,u)+(y,v)]. \quad \text{Q.E.D.}$$

PROOF. As in Lemma 1(b).

Since $Q(t,(x,u),A)$ is a transition function, we can define a nonnegative measurable function q on $(E,\mathcal{B})$ to be invariant with respect to Q if $Q_t q = q$ for all $t \geqslant 0$, where

$$Q_t q(x,u) = \int_E Q(t,(x,u),d(y,v))q(y,v) \qquad ((x,u) \in E).$$

For convenience we call such a function Q_t-invariant. We shall also define a minimal Q_t-invariant function to be a Q_t-invariant function q such that for every Q_t-invariant function r with $0 \leqslant r \leqslant q$ we have $r = cq$ for some constant c. We shall proceed to show that the constant function 1 is a minimal Q_t-invariant function. First we prove two lemmas.

LEMMA 7. *If r is a Q_t-invariant function, $r(0,0) < \infty$, then for each $u > 0$, $r(x,u)$ is a continuous function in x.*

PROOF. It is clear that r is Q_t-invariant if and only if rh is invariant. Hence the continuity of $r(x,u)$ in x for $u > 0$ follows from Lemma 2. Q.E.D.

Regarding the Markov process $Y = (y_t, \zeta, \mathfrak{M}_t, P_y)$ as a stochastic process $\{y_t : t \geqslant 0\}$ on $(E,\mathcal{B})$ with a probability space $(\Omega, \mathfrak{M}, P_{(0,0)})$, we prove

LEMMA 8. *The stochastic process $\{y_t : t \geqslant 0\}$ has stationary independent increments.*

PROOF. (a) The increments are stationary because for any $A \in \mathcal{B}$, $0 \leqslant s < t$,

$$\begin{aligned} &P_{(0,0)}\{y_t - y_s \in A\} \\ &\quad = \int_E Q(s,(0,0),d(y,v)) \int_E Q(t-s,(y,v),d(z,w)) \\ &\qquad \times \chi_{\{(z,w)-(y,v) \in A\}}((y,v),(z,w)), \end{aligned}$$

which by Lemma 6 is

$$= \int_E Q(s,(0,0),d(y,v)) \int_E Q(t-s,(0,0),d(z,w))\chi_A(z,w)$$
$$= Q(t-s,(0,0),A).$$

(b) The increments are independent because for any $A_1, \ldots, A_{n-1} \in \mathfrak{B}$, $0 \leqslant t_1 < t_2 < \cdots < t_n$,

$$P_{(0,0)}\{y_{t_2} - y_{t_1} \in A_1, y_{t_3} - y_{t_2} \in A_2, \ldots, y_{t_n} - y_{t_{n-1}} \in A_{n-1}\}$$
$$= \int_E Q(t_1,(0,0),d(z_1,w_1)) \int_E Q(t_2 - t_1,(z_1,w_1),d(z_2,w_2)) \int_E$$
$$\cdots \int_E Q(t_n - t_{n-1},(z_{n-1},w_{n-1}),d(z_n,w_n))$$
$$\times \chi_{\{(z_i,w_i)-(z_{i-1},w_{i-1}) \in A_{i-1}, i=2,\ldots,n\}}((z_1,w_1),\ldots,(z_n,w_n))$$

which by Lemma 6 is

$$= \int_E Q(t_1,(0,0),d(z_1,w_1)) \int_E Q(t_2 - t_1,(0,0),d(z_2,w_2)) \int_E$$
$$\cdots \int_E Q(t_n - t_{n-1},(0,0),d(z_n,w_n))\chi_{A_1}(z_2,w_2)\cdots\chi_{A_{n-1}}(z_n,w_n)$$
$$= Q(t_2 - t_1,(0,0),A_1)\cdots Q(t_n - t_{n-1},(0,0),A_{n-1})$$
$$= P_{(0,0)}\{y_{t_2} - y_{t_1} \in A_1\}\cdots P_{(0,0)}\{y_{t_n} - y_{t_{n-1}} \in A_{n-1}\}. \quad \text{Q.E.D.}$$

PROPOSITION 1. *The constant function* 1 *is a minimal* Q_t*-invariant function.*

PROOF. It is easy to see that 1 is Q_t-invariant. To show that 1 is also minimal we let r be another Q_t-invariant function on $(E,\mathfrak{B})$ such that $0 \leqslant r \leqslant 1$, we shall prove that r is a constant.

(a) The stochastic process $\{r(y_t): t \geqslant 0\}$ forms a martingale, i.e. (i) $E_{(0,0)}[r(y_t)] < \infty$ for all $t \geqslant 0$, and (ii) for any s, t $(0 \leqslant s < t)$,

$$E_{(0,0)}\{r(y_t)|r(y_u): u \leqslant s\} = r(y_s)$$

almost surely $[P_{(0,0)}]$. (i) follows from the boundedness of r. To prove (ii), let $\Delta \in \sigma\{r(y_u): u \leqslant s\}$; then

$$E_{(0,0)}[\chi_\Delta r(y_t)] = E_{(0,0)}[\chi_\Delta E_{y_s} r(y_{t-s})]$$
$$= E_{(0,0)}\left[\chi_\Delta \int_E Q(t-s,y_s,d(z,w))r(z,w)\right]$$
$$= E_{(0,0)}[\chi_\Delta r(y_s)]$$

because r is Q_t-invariant. The above implies that

$$E_{(0,0)}\{r(y_t)|r(y_u): u \leqslant s\} = r(y_s) \text{ a.s. } [P_{(0,0)}].$$

(b) The martingale convergence theorem implies that

$$r_\infty = \lim_{n\to\infty} r(y_n) = \lim_{n\to\infty} r\left[\sum_{i=1}^{n} (y_i - y_{i-1})\right]$$

exists almost surely $[P_{(0,0)}]$. For each number a, the event $\{r_\infty < a\}$ is in the tail σ-field of the random vectors $\{z_n: n = 1,2,\dots\}$, each of which is the partial sum of a sequence of independent identically distributed random vectors $\{y_i - y_{i-1}; i = 1,2,\dots\}$ (Lemma 8). By the Hewitt-Savage zero one law, the event $\{r_\infty < a\}$ has $P_{(0,0)}$-probability zero or one. Since a is arbitrary, $r_\infty = k$ (a constant) with $P_{(0,0)}$-probability one.

(c) r_∞ is a closing random variable for the martingale $\{r(y_n): n = 0,1,2,\dots\}$ because r is bounded. Thus

$$k = E_{(0,0)}\{r_\infty|r(y_0), r(y_1), \dots, r(y_n)\} = r(y_n) \quad \text{a.s. } [P_{(0,0)}]$$

for every $n = 0, 1, 2, \dots$. Hence $r(y,n) = k$ a.e. $[q(n,0,dy)]$ and, hence, a.e. with respect to Lebesgue measure on R. For any $(x,u) \in E$, let $n > u$ be any integer. Then

$$r(x,u) = \int_E Q(n-u,(x,u),d(y,v))r(y,v) = \int_R q(n-u,x,dy)r(y,n) = k$$

since $q(n-u,x,dy)$ has a density with respect to Lebesgue measure. Q.E.D.

Lastly, h is minimal because if g is another invariant function such that $0 \leqslant g \leqslant h$, then g/h is Q_t-invariant and hence $g = ch$ for some constant c by Proposition 1. This finishes the sufficiency part of Theorem 1.

4. **Proof of Theorem 2.** Theorem 2 is obtained by solving (a) and (b) of Theorem 1 for h. Actually we look for positive solutions, since by Lemma 1 the minimal invariant functions h which satisfy $h(0,0) = 1$ are positive. Q.E.D.

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Department of Mathematics, University of Singapore, Singapore 10, Republic of Singapore